On the error term concerning the number of subgroups of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ with $m, n \leq x$

by

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1. Introduction. Let \mathbb{Z}_m be the additive group of residue classes modulo m. For arbitrary positive integers m and n consider the group $G := \mathbb{Z}_m \times \mathbb{Z}_n$, which is isomorphic to $\mathbb{Z}_{gcd(m,n)} \times \mathbb{Z}_{lcm(m,n)}$. When gcd(m,n) = 1, G is cyclic and isomorphic to \mathbb{Z}_{mn} . When gcd(m,n) > 1, G has rank two. Let s(m,n) and c(m,n) denote the total number of subgroups and the number of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, respectively.

Here s(m,n) and c(m,n) are multiplicative functions of two variables, that is,

(1.1)
$$s(m,n) = \prod_{p} s(p^{\nu_{p}(m)}, p^{\nu_{p}(n)}),$$

(1.2)
$$c(m,n) = \prod_{p} c(p^{\nu_{p}(m)}, p^{\nu_{p}(n)}),$$

for all $m, n \in \mathbb{N}$. Furthermore, for the rank two *p*-group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ with $1 \leq a \leq b$, one has the following formulas:

(1.3)
$$s(p^a, p^b) = \frac{(b-a+1)p^{a+2} - (b-a-1)p^{a+1} - (a+b+3)p + (a+b+1)}{(p-1)^2}$$

and

(1.4)
$$c(p^a, p^b) = 2(1 + p + p^2 + \dots + p^{a-1}) + (b - a + 1)p^a.$$

Hence, one can compute s(m, n) and c(m, n) by using (1.1), (1.3) and (1.2), (1.4), respectively. However, the following more compact identities hold for all $m, n \in \mathbb{N}$:

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(1.5)
$$s(m,n) = \sum_{d|m,e|n} \gcd(d,e) = \sum_{d|\gcd(m,n)} \phi(d)\tau(m/d)\tau(n/d),$$

(1.6)
$$c(m,n) = \sum_{d|m,e|n} \phi(\gcd(d,e)) = \sum_{d|\gcd(m,n)} (\mu * \phi)(d)\tau(m/d)\tau(n/d).$$

For general properties of the subgroup lattice of finite abelian groups, see R. Schmidt [10] and M. Suzuki [11]. We note that formula (1.3) was deduced by using Goursat's lemma for groups in [3, 9], and using the concept of the fundamental group lattice in [12, 13]. Formula (1.4) was given in [13]. Identities (1.5) and (1.6) were derived in [4] by a simple elementary method, and in [14] by using Goursat's lemma for groups.

W. G. Nowak and L. Tóth [7] studied the average orders of the functions s(m,n) and c(m,n). Suppose x > 0 is a real number. Define

$$S^{(1)}(x) := \sum_{m,n \le x} s(m,n), \quad S^{(2)}(x) := \sum_{\substack{m,n \le x \\ \gcd(m,n) > 1}} s(m,n),$$
$$S^{(3)}(x) := \sum_{m,n \le x} c(m,n), \quad S^{(4)}(x) := \sum_{\substack{m,n \le x \\ \gcd(m,n) > 1}} c(m,n).$$

Here $S^{(2)}(x)$ and $S^{(4)}(x)$ denote the total number of subgroups and cyclic subgroups, respectively, of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ having rank two, with $m, n \leq x$.

W. G. Nowak and L. Tóth [7] proved that for every j with $1 \le j \le 4$,

(1.7)
$$S^{(j)}(x) = x^2 \sum_{r=0}^{3} A_{j,r} \log^r x + O(x^{1117/701+\varepsilon}),$$

where $A_{j,r}$ $(1 \le j \le 4, 0 \le r \le 3)$ are explicit constants, whose definitions are omitted here. Note that 1117/701 = 1.593437... In fact, the error term in (1.7) is $O(x^{(3-\theta)/(2-\theta)+\varepsilon})$, where θ is the exponent in the Dirichlet divisor problem for $\tau(n)$. The exponent 1117/701 is obtained from $\theta = 131/416$ of M. N. Huxley [5]. The asymptotic formula (1.7) holds for the slightly better exponent 4427/2779 = 1.593019... by using the exponent $\theta = 517/1648$ obtained in [2]. Note that the limit of this approach is 11/7 = 1.571428...with $\theta = 1/4$.

In this paper we shall prove the following theorem, which improves the above error terms.

THEOREM 1.1. The asymptotic formulas

$$S^{(j)}(x) = x^2 \sum_{r=0}^{3} A_{j,r} \log^r x + O(x^{3/2} (\log x)^{6.5})$$

hold for every j with $1 \le j \le 4$.

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For the proof we use a multidimensional Perron formula and the complex integration method.

NOTATION. Throughout this paper, \mathbb{N} denotes the set of all positive integers, ϕ is Euler's totient function, μ is the Möbius function, ζ denotes the Riemann zeta-function, and $\tau_k(n)$ denotes the number of ways n can be written as a product of k positive integers ($\tau(n) = \tau_2(n)$). Let $n = \prod_p p^{\nu_p(n)}$ denote the prime power factorization of $n \geq 2$, where the product is over the primes p and all but a finite number of the exponents $\nu_p(n)$ are zero.

2. Preliminary lemmas

LEMMA 2.1 ([7]). Suppose $\Re z, \Re w > 1$. Then

$$S(z,w) := \sum_{m \ge 1} \sum_{n \ge 1} \frac{s(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)\zeta^{-1}(z+w),$$

$$C(z,w) := \sum_{m \ge 1} \sum_{n \ge 1} \frac{c(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)\zeta^{-2}(z+w).$$

LEMMA 2.2. Suppose that $r \geq 2$ is a fixed integer and $f(n_1, \ldots, n_r)$ is an arithmetical function of r variables that is symmetric in n_1, \ldots, n_r and whose Dirichlet series

$$F(z_1, \dots, z_r) := \sum_{n_1 \ge 1} \dots \sum_{n_r \ge 1} \frac{f(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}}$$

is absolutely convergent for $\Re z_j > \sigma_a$ $(1 \le j \le r)$ with some $\sigma_a > 0$. Suppose $x, T \ge 10$ are two parameters, and define

$$b = \sigma_a + \frac{1}{\log x}, \quad T_j = 2^{j-1}T \quad (1 \le j \le r).$$

Then

$$\sum_{n_1 \le x} \cdots \sum_{n_r \le x} f(n_1, \dots, n_r) h\left(\frac{x}{n_1}\right) \cdots h\left(\frac{x}{n_r}\right)$$
$$= \frac{1}{(2\pi i)^r} \int_{b-iT_1}^{b+iT_1} \cdots \int_{b-iT_r}^{b+iT_r} F(z_1, \dots, z_r) x^{z_1 + \dots + z_r} \frac{dz_r \cdots dz_1}{z_r \cdots z_1} + O(x^{r\sigma_a} E_f(x, T)),$$

where

$$E_f(x,T) := \sum_{n_1 \ge 1} \cdots \sum_{n_r \ge 1} \frac{|f(n_1, \dots, n_r)| (n_1 \cdots n_r)^{-\sigma_a - 1/\log x}}{\min_{1 \le j \le r} T |\log(x/n_j)| + 1}$$

and

$$h(y) := \begin{cases} 1 & \text{if } y > 1, \\ 1/2 & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

Proof. This is a multiple type Perron formula, which easily follows from [1, Propositions 5 and 6].

LEMMA 2.3. Suppose $\ell = 0$ or $\ell = 1$. For $\sigma > 1$ we have

$$\begin{aligned} \zeta^{(\ell)}(\sigma+it) &\ll \min\left(\frac{1}{(\sigma-1)^{1+\ell}}, \log^{1+\ell}(|t|+2)\right), \\ \zeta^{-1}(\sigma+it) &\ll \min\left(\frac{1}{\sigma-1}, \log(|t|+2)\right). \end{aligned}$$

Proof. The first estimate for $\ell = 0$ can be found in Pan and Pan [8, Chapter 7]. The first estimate for $\ell = 1$ follows from the result for $\ell = 0$ and Cauchy's theorem. The second estimate can also be found in Pan and Pan [8, Chapter 7].

LEMMA 2.4. Suppose $\ell = 0$ or $\ell = 1$. Then for $1/2 \le \sigma \le 1$ we have $\zeta^{(\ell)}(\sigma + it) \ll (|t| + 2)^{(1-\sigma)/3} \log^{1+\ell}(|t| + 2).$

Proof. The estimate for $\ell = 0$ follows from the bounds

$$\zeta(1/2 + it) \ll (|t| + 2)^{1/6},$$

$$\zeta(1 + it) \ll \log(|t| + 2)$$

and the Phragmén–Lindelöf principle. The estimate for $\ell = 1$ follows from the result for $\ell = 0$ and Cauchy's theorem.

LEMMA 2.5. Suppose V>10 is a large parameter and $|u-1/2|\leq 1/\log V.$ Then

(2.1)
$$\int_{-V}^{V} |\zeta(u+iv)|^4 \, dv \ll V \log^4 V,$$

(2.2)
$$\int_{-V}^{V} |\zeta(u+iv)|^2 dv \ll V \log V,$$

(2.3)
$$\int_{-V}^{V} |\zeta'(1/2 + iv)|^2 \, dv \ll V \log^3 V,$$

(2.4)
$$\int_{-V}^{V} |\zeta(u+iv)|^2 \, dv \ll V \quad (0.6 < u < 2).$$

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Proof. The estimates (2.1) and (2.2) can be found in Pan and Pan [8, Chapter 25]. The estimate (2.3) follows from (2.2) and Cauchy's theorem. Actually, (2.4) holds for $u > 1/2 + \varepsilon$: see for example Ivić [6, (8.112)].

3. Proof of Theorem 1.1. We only prove the theorem for the function s(m,n), i.e., for the sums $S^{(1)}(x)$ and $S^{(2)}(x)$. The proof for c(m,n) is similar.

By Lemmas 2.1 and 2.2 with r = 2 and $\sigma_a = 1$ we have

(3.1)
$$\sum_{m \le x} \sum_{n \le x} s(m,n) h\left(\frac{x}{m}\right) h\left(\frac{x}{n}\right) = I(x,T) + O(x^2 E(x,T)),$$

where

$$I(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} \int_{b-2iT}^{b+2iT} \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)x^{z+w}}{\zeta(z+w)wz} \, dw \, dz,$$
$$E(x,T) := \sum_{n\geq 1} \sum_{m\geq 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1} + \sum_{n\geq 1} \sum_{m\geq 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/m)|+1},$$

and T is a parameter to be determined such that $10 \le T \le x/2$.

3.1. Estimate of E(x,T). Since s(m,n) is symmetric, that is, s(m,n) = s(n,m), we have

(3.2)
$$E(x,T) = 2\sum_{n\geq 1}\sum_{m\geq 1}\frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1}.$$

Write

$$\sum_{n \ge 1} \sum_{m \ge 1} \frac{s(m, n)(mn)^{-1 - 1/\log x}}{T |\log(x/n)| + 1} = E_1 + E_2 + E_3,$$

where

$$E_1 := \sum_{n \le x/2} \sum_{m \ge 1} \frac{s(m, n)(mn)^{-1 - 1/\log x}}{T|\log(x/n)| + 1},$$
$$E_2 := \sum_{x/2 < n \le 2x} \sum_{m \ge 1} \frac{s(m, n)(mn)^{-1 - 1/\log x}}{T|\log(x/n)| + 1},$$
$$E_3 := \sum_{n > 2x} \sum_{m \ge 1} \frac{s(m, n)(mn)^{-1 - 1/\log x}}{T|\log(x/n)| + 1}.$$

If $n \le x/2$ or n > 2x then $|\log(x/n)| \gg 1$, so by Lemma 2.1 (with $z = w = 1 + 1/\log x$) and Lemma 2.3 we have

(3.3)
$$E_1 + E_3 \ll T^{-1} \sum_{n \ge 1} \sum_{m \ge 1} s(m, n) (mn)^{-1 - 1/\log x}$$
$$= T^{-1} \zeta^4 (1 + 1/\log x) \zeta (1 + 2/\log x) \zeta^{-1} (2 + 2/\log x)$$
$$\ll T^{-1} \log^5 x.$$

So it suffices to bound E_2 . We have

(3.4)
$$E_2 \ll \sum_{x/2 < n \le 2x} \frac{n^{-1}}{T |\log(x/n)| + 1} \sum_{m \ge 1} s(m, n) m^{-1 - 1/\log x}.$$

Recall that $b = 1 + 1/\log x$. From (1.5) we have

$$\sum_{m \ge 1} s(m,n)m^{-b} = \sum_{m \ge 1} \sum_{d|m,e|n} \gcd(d,e)m^{-b} = \sum_{m \ge 1} m^{-b} \sum_{dm_1=m,en_1=n} \gcd(d,e)$$
$$\leq \sum_{m \ge 1} m^{-b} \sum_{\varrho d_1m_1=m,\,\varrho e_1n_1=n} \varrho = \sum_{\varrho e_1n_1=n} \varrho \sum_{\varrho d_1m_1=m} (\varrho d_1m_1)^{-b}$$
$$= \sum_{\varrho e_1n_1=n} \varrho^{1-b} \sum_{d_1m_1=m} (d_1m_1)^{-b} \ll \zeta^2(b)\tau_3(n) \ll \tau_3(n)\log^2 x.$$

Inserting this estimate into (3.4) and noting that $\tau_3(n) \ll n^{\varepsilon}$ we have

(3.5)
$$E_2 \ll \frac{\log^2 x}{x} \sum_{x/2 < n \le 2x} \frac{\tau_3(n)}{T |\log(x/n)| + 1} \ll \frac{x^{\varepsilon}}{x} (E_{21} + E_{22} + E_{23}),$$

say, where

$$E_{21} := \sum_{x/2 < n \le xe^{-1/T}} \frac{1}{T |\log(x/n)| + 1},$$

$$E_{22} := \sum_{xe^{-1/T} < n \le xe^{1/T}} \frac{1}{T |\log(x/n)| + 1},$$

$$E_{23} := \sum_{xe^{1/T} < n \le 2x} \frac{1}{T |\log(x/n)| + 1}.$$

For E_{22} we have

$$E_{22} \ll \sum_{xe^{-1/T} < n \le xe^{1/T}} 1 \ll xe^{1/T} - xe^{-1/T} + 1 \ll x/T.$$

For E_{21} we have

$$E_{21} \ll \frac{1}{T} \sum_{x/2 < n \le xe^{-1/T}} \frac{1}{|\log(x/n)|}$$

$$\ll \frac{1}{T} \sum_{[x]-xe^{-1/T} \le k \le x/2} \frac{1}{|\log(x/([x]-k))|} \quad (n = [x]-k)$$

$$\ll \frac{1}{T} \sum_{x/T \ll k \le x/2} \frac{1}{\log x - \log([x]-k)} \ll \frac{1}{T} \sum_{x/T \ll k \le x/2} \frac{x}{k} \ll \frac{x \log x}{T}$$

Similarly, we deduce

$$E_{23} \ll \frac{x \log x}{T}.$$

Inserting the above estimates into (3.5) we conclude that

(3.6)
$$E_2 \ll \frac{x^{\varepsilon}}{T}.$$

From (3.2), (3.3) and (3.6) we get the following proposition.

Proposition 3.1. If $10 \le T \le x/2$, then

 $E(x,T) \ll x^{\varepsilon}T^{-1}.$

3.2. Evaluation of the integral I(x,T) for the variable w. Consider the rectangle domain formed by the four points $w = b \pm 2iT$, $w = 1/2 \pm 2iT$. In this domain the integrand

(3.7)
$$g(z,w) := \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)x^{z+w}}{\zeta(z+w)wz}$$

has two poles, namely w = 1, which is a pole of order 2, and w = 2 - z, a simple pole. By the residue theorem we get

(3.8)
$$I(x,T) = J_1(x,T) + J_2(x,T) + H_1(x,T) + H_2(x,T) - H_3(x,T),$$

where

$$\begin{split} J_1(x,T) &:= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \operatorname{Res}_{w=1} g(z,w) \, dz, \\ J_2(x,T) &:= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \operatorname{Res}_{w=2-z} g(z,w) \, dz, \\ H_1(x,T) &:= \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} dz \int_{1/2+2iT}^{b+2iT} g(z,w) \, dw, \end{split}$$

$$H_2(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} dz \int_{1/2-2iT}^{1/2+2iT} g(z,w) \, dw,$$

$$H_3(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} dz \int_{1/2-2iT}^{b-2iT} g(z,w) \, dw.$$

We estimate $H_1(x,T)$ first. In this case by Lemmas 2.3 and 2.4 we have (noting that $|t| \leq T$), uniformly for $1/2 \leq u \leq b = 1 + 1/\log x$,

$$\begin{split} g(z,w) &= g(b+it,u+2iT) \\ &\ll \frac{|\zeta(b+it)|^2}{|t|+1} \frac{x^{b+u}}{T} |\zeta^2(u+2iT)| \left| \zeta(u+1/\log x+i(t+2T)) \right| \\ &\ll \frac{\log^2 x}{|t|+1} \times \frac{x^{1+u}}{T} T^{\max(1-u,0)} \log^3 T \\ &\ll \frac{x \log^5 x}{T} \times \frac{1}{|t|+1} \times x^u T^{\max(1-u,0)}. \end{split}$$

So we get

(3.9)
$$H_1(x,T) \ll \frac{x \log^5 x}{T} \int_{-T}^{T} \frac{1}{|t|+1} dt \left(\int_{1/2}^{1} x^u T^{1-u} du + \int_{1}^{b} x^u du \right) \ll \frac{x^2 \log^6 x}{T}.$$

Similarly, we have

(3.10)
$$H_3(x,T) \ll \frac{x^2 \log^6 x}{T}.$$

Now we estimate $H_2(x,T)$. In this case by Lemmas 2.3 and 2.4 we have, with $|t| \leq T$, $|v| \leq 2T$,

$$\begin{split} g(z,w) &= g(b+it,1/2+iv) \\ &\ll \frac{|\zeta(b+it)|^2}{|t|+1} \frac{x^{b+1/2}}{|v|+1} |\zeta^2(1/2+iv)| \left| \zeta(1/2+1/\log x+i(t+v)) \right| \\ &\ll x^{3/2} \log^2 x \times \frac{|\zeta(1/2+iv)|^2 |\zeta(1/2+1/\log x+i(t+v))|}{(|t|+1)(|v|+1)}. \end{split}$$

Hence

$$\begin{array}{ll} (3.11) & H_2(x,T) \\ & \ll x^{3/2} \log^2 x \int\limits_{-T}^{T} dt \int\limits_{-2T}^{2T} \frac{|\zeta(1/2+iv)|^2 |\zeta(1/2+1/\log x+i(t+v))|}{(|t|+1)(|v|+1)} \, dv \\ & \ll x^{3/2} \log^2 x \left(\int_1 + \int_2\right), \\ \text{where } \int_1 = \int_{|v| \le |t|} \text{ and } \int_2 = \int_{|t| \le |v|}. \end{array}$$

We first estimate \int_1 . Let $L_1(v) := \int_0^v |\zeta(1/2+iy)|^2 dy$. By (2.2) and partial summation we get

$$(3.12) \qquad \int_{-V}^{V} \frac{|\zeta(1/2+iv)|^2}{|v|+1} dv \\ \ll 1 + \int_{1}^{V} \frac{|\zeta(1/2+iv)|^2}{v} dv = 1 + \int_{1}^{V} \frac{dL_1(v)}{v} \\ \ll 1 + \frac{L_1(V)}{V} + \int_{1}^{V} \frac{L_1(v)}{v^2} dv \ll \log V + \int_{1}^{V} \frac{\log v}{v} dv \ll \log^2 V.$$

From the estimate (2.2) and Cauchy's inequality we get

$$\int_{0}^{T} |\zeta(1/2 + 1/\log x + iy)| \, dy \ll T \log^{1/2} T,$$

which by partial summation yields

(3.13)
$$\int_{-2T}^{2T} \frac{|\zeta(1/2 + 1/\log x + iy)| \, dy}{|y| + 1} \ll \log^{1.5} T.$$

Note that in \int_1 we have

$$|t + v| + 1 \le |t| + |v| + 1 \le 2(|t| + 1).$$

Thus from (3.12) and (3.13) we get

$$\begin{split} \int_{1} &= \int_{|v| \leq |t|} \frac{|\zeta(1/2 + iv)|^{2} |\zeta(1/2 + 1/\log x + i(t + v))|}{(|v| + 1)(|t + v| + 1)} \times \frac{|t + v| + 1}{|t| + 1} \, dv \, dt \\ &\leq 2 \int_{-T}^{T} \frac{|\zeta(1/2 + iv)|^{2}}{|v| + 1} \, dv \int_{|v| \leq |t|} \frac{|\zeta(1/2 + 1/\log x + i(t + v))|}{|t + v| + 1} \, dt \\ &\leq 2 \int_{-T}^{T} \frac{|\zeta(1/2 + iv)|^{2}}{|v| + 1} \, dv \int_{-2T}^{2T} \frac{|\zeta(1/2 + 1/\log x + iy)|}{|y| + 1} \, dy \\ &\ll (\log T)^{3.5} \ll (\log x)^{3.5}. \end{split}$$

Now we estimate \int_2 . Similar to (3.12), by (2.1) and (2.2) we get (3.15) $V = |\zeta(1/2 + iv)|^4$

$$\int_{-V}^{V} \frac{|\zeta(1/2+iv)|^4}{|v|+1} \, dv \ll \log^5 V, \qquad \int_{-V}^{V} \frac{|\zeta(1/2+1/\log x+iv)|^2}{|v|+1} \, dv \ll \log^2 V.$$

Note that in \int_2 ,

$$|t+v|+1 \le |t|+|v|+1 \le 2(|v|+1)$$
, so $\frac{1}{|v|+1} \le \frac{2}{|t+v|+1}$.

Thus via (3.15) we obtain

$$\begin{split} \int_{2} &= \int_{|t| \leq |v|} \frac{|\zeta(1/2 + iv)|^{2} |\zeta(1/2 + 1/\log x + i(t + v))|}{(|t| + 1)(|v| + 1)} \, dv \, dt \\ &\leq \int_{-T}^{T} \frac{dt}{|t| + 1} \int_{|t| \leq |v| \leq 2T} \frac{|\zeta(1/2 + iv)|^{2} |\zeta(1/2 + 1/\log x + i(t + v))|}{|v| + 1} \, dv \\ &\leq 2 \int_{-T}^{T} \frac{dt}{|t| + 1} \int_{|t| \leq |v| \leq 2T} \frac{|\zeta(1/2 + iv)|^{2}}{(|v| + 1)^{1/2}} \times \frac{|\zeta(1/2 + 1/\log x + i(t + v))|}{(|t + v| + 1)^{1/2}} \, dv \\ &\leq 2 \int_{-T}^{T} \frac{dt}{|t| + 1} \int_{-2T}^{2T} \frac{|\zeta(1/2 + iv)|^{2}}{(|v| + 1)^{1/2}} \times \frac{|\zeta(1/2 + 1/\log x + i(t + v))|}{(|t + v| + 1)^{1/2}} \, dv \\ &\ll \int_{-T}^{T} \frac{dt}{|t| + 1} \left(\int_{-2T}^{2T} \frac{|\zeta(1/2 + iv)|^{4}}{|v| + 1} \, dv \right)^{1/2} \\ &\qquad \times \left(\int_{-2T}^{2T} \frac{|\zeta(1/2 + 1/\log x + i(t + v))|^{2}}{|t + v| + 1} \, dv \right)^{1/2} \\ &\ll \int_{-T}^{T} \frac{dt}{|t| + 1} \left(\int_{-2T}^{2T} \frac{|\zeta(1/2 + iv)|^{4}}{|v| + 1} \, dv \right)^{1/2} \left(\int_{-3T}^{3T} \frac{|\zeta(1/2 + 1/\log x + iy)|^{2}}{|y| + 1} \, dv \right)^{1/2} \\ &\ll (\log T)^{4.5} \ll (\log x)^{4.5}, \end{split}$$

which combined with (3.14) and (3.11) gives

(3.16)
$$H_2(x,T) \ll x^{3/2} (\log x)^{6.5}.$$

Now we evaluate $J_2(x,T)$. Since w = 2 - z is a simple pole of g(z,w), we have

$$\operatorname{Res}_{w=2-z} g(z,w) = \frac{x^2}{\zeta(2)} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}.$$

From (2.4) by partial summation we get

(3.17)
$$\int_{T}^{\infty} \frac{|\zeta(u+iv)|^2 \, dv}{v^2} \ll T^{-1}, \quad 0.6 < u < 2.$$

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So from (3.17) with $u = 1 - 1/\log x$ and Lemma 2.3 we have

$$\begin{split} J_2(x,T) &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, dz \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, dz \\ &+ O\left(x^2 \int_T^{\infty} \left| \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \right| | \, dz | \right) \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, dz \\ &+ O\left(x^2 \int_T^{\infty} \left| \frac{\zeta^2(1+1/\log x+it)\zeta^2(1-1/\log x-it)}{t^2} \right| \, dt \right) \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, ds + O(x^2T^{-1}\log^2 x). \end{split}$$

We shall show that the integral in the last line is a constant. By the residue theorem we have

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, dz = \frac{1}{2\pi i} \int_{4/3-i\infty}^{4/3+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \, dz.$$

By (3.17) with u = 2/3 we see that the integral on the right-hand side is absolutely convergent. Hence

(3.18)
$$J_2(x,T) = Cx^2 + O(x^2T^{-1}\log^2 x),$$

where C is an absolute constant.

Finally, we evaluate $J_1(x, T)$. We shall use the following easy fact: if G(s) is analytic at s = 1, then

(3.19)
$$\operatorname{Res}_{s=1} \zeta^2(s) G(s) = G'(1) + 2\gamma G(1),$$

where γ is the Euler constant.

Define

$$G_z(w) := \frac{\zeta(z+w-1)x^w}{\zeta(z+w)w}.$$

It is easy to see that

(3.20)
$$G'_{z}(w) = \frac{\zeta'(z+w-1)x^{w}}{\zeta(z+w)w} + \frac{\zeta(z+w-1)x^{w}\log x}{\zeta(z+w)w} - \frac{\zeta(z+w-1)\zeta'(z+w)x^{w}}{\zeta^{2}(z+w)w} - \frac{\zeta(z+w-1)x^{w}}{\zeta(z+w)w^{2}}.$$

From (3.19) and (3.20) we have

(3.21)
$$\operatorname{Res}_{w=1} g(z,w) = \frac{\zeta^2(z)x^z}{z} (G'_z(1) + 2\gamma G_z(1)) = \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} + \frac{\zeta^3(z)x^{z+1}\log x}{z\zeta(z+1)} + h(z)\frac{\zeta^3(z)x^{z+1}}{z},$$

where

$$h(z) := \frac{2\gamma}{\zeta(z+1)} - \frac{1}{\zeta(z+1)} - \frac{\zeta'(z+1)}{\zeta^2(z+1)}.$$

From (3.21) we have

(3.22)
$$J_1(x,T) = J_{11}(x,T) + J_{12}(x,T) + J_{13}(x,T),$$

where

$$J_{11}(x,T) := \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} dz,$$

$$J_{12}(x,T) := \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta^3(z)x^{z+1}\log x}{z\zeta(z+1)} dz,$$

$$J_{13}(x,T) := \frac{1}{2\pi i} \int_{b-iT}^{b+iT} h(z) \frac{\zeta^3(z)x^{z+1}}{z} dz.$$

By the residue theorem, we have

(3.23)
$$J_{11}(x,T) = \operatorname{Res}_{z=1} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} + L_1(x,T) + L_2(x,T) - L_3(x,T),$$

where

$$L_1(x,T) := \frac{1}{2\pi i} \int_{1/2+iT}^{b+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} dz,$$

$$L_2(x,T) := \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} dz,$$

$$L_3(x,T) := \frac{1}{2\pi i} \int_{1/2-iT}^{b-iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} dz.$$

Similar to the estimate for $H_1(x,T)$, we have

(3.24)
$$L_1(x,T) \ll \frac{x^2 \log^4 x}{T}, \quad L_3(x,T) \ll \frac{x^2 \log^4 x}{T}.$$

For $L_2(x,T)$, by (2.1) and (2.3), Cauchy's inequality and partial summation we obtain

$$(3.25) \quad L_2(x,T) \ll x^{3/2} \int_{-T}^{T} \frac{|\zeta(1/2+it)|^2 |\zeta'(1/2+it)|}{|t|+1} dt$$
$$\ll x^{3/2} \left(\int_{-T}^{T} \frac{|\zeta(1/2+it)|^4}{|t|+1} dt \right)^{1/2} \left(\int_{-T}^{T} \frac{|\zeta'(1/2+it)|^2}{|t|+1} dt \right)^{1/2}$$
$$\ll x^{3/2} \log^{4.5} T \ll x^{3/2} \log^{4.5} x.$$

Since z = 1 is the pole of $\zeta^2(z)\zeta'(z)$ of degree 4, we have

$$\operatorname{Res}_{z=1} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} = x^2 \sum_{j=0}^3 a_j \log^j x_j$$

where a_j are computable constants. So from (3.23)–(3.25) we get

(3.26)
$$J_{11}(x,T) = x^2 \sum_{j=0}^{3} a_j \log^j x + O\left(\frac{x^2 \log^4 x}{T} + x^{3/2} \log^{4.5} x\right).$$

Similarly,

(3.27)
$$J_{12}(x,T) = x^2 \sum_{j=0}^{3} b_j \log^j x + O\left(\frac{x^2 \log^4 x}{T} + x^{3/2} \log^{4.5} x\right),$$

(3.28)
$$J_{13}(x,T) = x^2 \sum_{j=0}^{3} c_j \log^j x + O\left(\frac{x^2 \log^4 x}{T} + x^{3/2} \log^{4.5} x\right),$$

where b_j and c_j are constants such that $b_0 = c_3 = 0$.

From (3.1), (3.8)–(3.10), (3.12), (3.16), (3.18), (3.22), (3.26)–(3.28) and Proposition 3.1 we get

(3.29)
$$\sum_{m,n \le x} s(m,n)h\left(\frac{x}{m}\right)h\left(\frac{x}{n}\right)$$
$$= x^2 \sum_{r=0}^3 A_{1,r} \log^r x + O\left(\frac{x^{2+\varepsilon}}{T} + x^{3/2} \log^{6.5} x\right)$$
$$= x^2 \sum_{r=0}^3 A_{1,r} \log^r x + O(x^{3/2} \log^{6.5} x)$$

by choosing T = x/4, where

$$A_{1,r} = a_r + b_r + c_r$$
 $(r = 1, 2, 3),$ $A_{1,0} = a_0 + b_0 + c_0 + C.$

3.3. Completion of proof. Suppose $N \ge 10$ is an integer. We shall give an upper bound of the sum

$$\sum_{m \le N} s(m, N).$$

By (1.5) we have

$$\begin{split} \sum_{m \leq N} s(m,N) &= \sum_{m \leq N} \sum_{d \mid m, e \mid N} \gcd(d,e) = \sum_{m \leq N} \sum_{\varrho d_1 m_1 = m, \, \varrho e_1 n_1 = N} \varrho \\ &= \sum_{\varrho e_1 n_1 = N} \varrho \sum_{\varrho d_1 m_1 \leq N} 1 = \sum_{\varrho e_1 n_1 = N} \varrho \sum_{e_1 m_1 \leq N/\varrho} 1 \\ &\ll \sum_{\varrho e_1 n_1 = N} \varrho \times \frac{N}{\varrho} \log \frac{N}{\varrho} \ll N \tau_3(N) \log N \ll N^{1+\varepsilon}. \end{split}$$

If x > 1 is not an integer, then

$$\sum_{m,n \le x} s(m,n) = \sum_{m,n \le x} s(m,n) h\left(\frac{x}{m}\right) h\left(\frac{x}{n}\right)$$

If x > 1 is an integer, then

$$\sum_{m,n \le x} s(m,n) = \sum_{m,n \le x} s(m,n)h\left(\frac{x}{m}\right)h\left(\frac{x}{n}\right) + \frac{1}{2}\sum_{n \le x} s(x,n) + \frac{1}{2}\sum_{m \le x} s(m,x) - s(x,x)/4.$$

From the above three formulas we see that for any x > 1,

$$\sum_{m,n \le x} s(m,n) = \sum_{m,n \le x} s(m,n)h\left(\frac{x}{m}\right)h\left(\frac{x}{n}\right) + O(x^{1+\varepsilon}),$$

which combined with (3.29) completes the proof of the asymptotic formula for $S^{(1)}(x) = \sum_{m,n \leq x} s(m,n)$.

Furthermore, from (1.5) we have

(3.30)
$$S^{(2)}(x) = S^{(1)}(x) - U(x)$$
, where $U(x) := \sum_{\substack{m,n \le x \\ \gcd(m,n)=1}} \tau(m)\tau(n)$.

From [7, Lemma 3.3] we have

(3.31)
$$U(x) = x^2 (b_2 \log^2 x + b_1 \log x + b_0) + O(x^{4/3 + \varepsilon}),$$

where b_j (j = 0, 1, 2) are explicit constants.

Now the required asymptotic formula for $S^{(2)}(x)$ follows from (3.30), (3.31) and our result for $S^{(1)}(x)$.

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