Ramanujan sums

László Tóth

http://www.ttk.pte.hu/matek/ltoth

Institute of Mathematics and Informatics
University of Pécs
Hungary

May 2007

Dumlupinar University, Kütahya
Turkey

1. Periodic functions

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called **periodic** if there is a number $r \in \mathbb{N}$ such that $f(n + r) = f(n)$ for all $n \in \mathbb{N}$. This is equivalent to the condition: $f(n_1) = f(n_2)$ for all $n_1, n_2 \in \mathbb{N}, n_1 \equiv n_2 \pmod{r}$. Here r is called the **period** of the function f and we say that f is periodic $\pmod{r}$.

If f is periodic $\pmod{r}$, then it can be extended to a function defined on $\mathbb{Z}$, denoted also by f, such that $f(n_1) = f(n_2)$ for every $n_1, n_2 \in \mathbb{Z}$, $n_1 \equiv n_2 \pmod{r}$.

Examples. 1) The function $f(n) = (n, r)$, where (n, r) is the gcd of n and r, is periodic $\pmod{r}$.

2) $e(n) := \exp(2\pi i n/r) = \cos(2\pi n/r) + i \sin(2\pi n/r)$ is periodic $\pmod{r}$. Here the values of $e(n)$ are the roots of unity of order r. More generally, $e_k(n) := \exp(2\pi i k n/r)$ is also periodic $\pmod{r}$, where $k \in \mathbb{N}$ is fixed and $e_k(n)$ are the k-th powers of the roots of unity of order r.

3) The function $c_r(n) := \sum_{\substack{k \pmod{r} \\ (k,r)=1}} \exp(2\pi i k n/r)$, called **Ramanujan sum**, is also periodic $\pmod{r}$, here the sum is over a reduced residue system $\pmod{r}$.

If r is a period of the function f, then each multiple of r is also a period. There exists a least (positive) period. What is the set of all periods?

The answer is given by

Theorem 1.1. *If r_1 and r_2 are periods of the function f, then the gcd (r_1, r_2) is also a period. The least period divides each period, therefore the set of periods is the set of multiples of the least period.*

Proof. There exist $u, v \in \mathbb{Z}$ such that $(r_1, r_2) = ur_1 + vr_2$. Using that r_1 and r_2 are periods,

$$f(n + (r_1, r_2)) = f(n + ur_1 + vr_2) = f(n + ur_1) = f(n)$$

for all n. If r_0 is the least period and r is an other period, then (r_0, r) is also one, and since $r_0 \leq (r_0, r)$ we obtain $r_0 \mid r$. ■

Theorem 1.2.

$$\sum_{k \pmod r} e_k(n) = \sum_{k \pmod r} \exp(2\pi i k n/r) = \begin{cases} r, & \text{if } r \mid n, \\ 0, & \text{if } r \nmid n. \end{cases}$$

Proof. Let $\varepsilon = \exp(2\pi i n/r)$. By the formula for the sum of geometric sequences,

$$\sum_{k \pmod r} \exp(2\pi i k n/r) = \sum_{k=0}^{r-1} \varepsilon^k = \begin{cases} r, & \text{if } \varepsilon = 1, \\ \frac{\varepsilon^r - 1}{\varepsilon - 1} = 0, & \text{if } \varepsilon \neq 1, \end{cases}$$

where $\varepsilon = 1$ iff $r \mid n$. ■

Remark. A usual notation is $e(x) := \exp(2\pi i x)$, $x \in \mathbb{Q}$. If $a \equiv b \pmod r$, then $e(\frac{a}{r}) = e(\frac{b}{r})$, and this is the property that gives number theoretic importance to certain trigonometric sums, involving $e(x)$.

We show that every periodic function (mod r) can be written as a sum of the functions $e_k(n) = \exp(2\pi i k n/r)$. More exactly,

Theorem 1.3. *If the function f is periodic (mod r), then f can be written as*

$$f(n) = \sum_{k=1}^{r} g(k) \exp(2\pi i k n/r), \quad n \in \mathbb{N}, \tag{1}$$

where the values $g(k)$ are unique and are given by

$$g(k) = \frac{1}{r} \sum_{j=1}^{r} f(j) \exp(-2\pi i j k/r), \quad 1 \le k \le r. \tag{2}$$

Proof. For all $n \in \mathbb{N}$, $F(n) := \sum_{k=1}^{r} g(k) \exp(2\pi i k n/r) =$

$$= \sum_{k=1}^{r} \frac{1}{r} \left(\sum_{j=1}^{r} f(j) \exp(-2\pi i j k/r) \right) \exp(2\pi i k n/r) =$$

$$= \frac{1}{r} \sum_{j=1}^{r} f(j) \sum_{k=1}^{r} \exp(2\pi i k(n-j)/r).$$

By Theorem 1.2. the inner sum is r if $r \mid (n-j) \Leftrightarrow j \equiv n \pmod{r}$ and otherwise it is 0. Hence $F(n) = f(n)$.

Now suppose that $f(n)$ van be written in the form (1) and also in this way:

$$f(n) = \sum_{k=1}^{r} g'(k) \exp(2\pi ikn/r), \quad n \in \mathbb{N}. \tag{3}$$

We show that $g(j) = g'(j)$ for all j. By (1) and (3)

$$\sum_{k=1}^{r} (g(k) - g'(k)) \exp(2\pi ikn/r) = 0, \quad n \in \mathbb{N}.$$

Multiplying by $\exp(-2\pi ijn/r)$ and summing we have:

$$\sum_{n=1}^{r} \sum_{k=1}^{r} (g(k) - g'(k)) \exp(2\pi in(k-j)/r) = 0,$$

$$\sum_{k=1}^{r} (g(k) - g'(k)) \sum_{n=1}^{r} \exp(2\pi in(k-j)/r) = 0,$$

and using again Theorem 1.2 we obtain $(g(j) - g'(j))r = 0$, consequently $g(j) = g'(j)$. ■

Remarks. 1. Formula (1) is called the **finite Fourier expansion** of f and the values $g(k)$ given by (2) are the **Fourier coefficients** of f.

2. The function g given by (2) is also periodic (mod r).

3. Let $\mathcal{P}_r$ denote the set of periodic functions (mod r), which is a complex linear space with the addition of functions and pointwise multiplication. Moreover, $\mathcal{P}_r$ is isomorphic to the euclidean space $\mathbb{C}^r$, hence its dimension is r and the usual inner product of $\mathbb{C}^r$ can be written here in this form:

$$\langle f, g\rangle = \frac{1}{r} \sum_{k \pmod r} f(k)\overline{g(k)},$$

where the sum is over a complete system of residues (mod r) and $\overline{g(k)}$ is the complex conjugate of $g(k)$.

According to Theorem 1.2., the functions e_j, $1 \le j \le r$, given by $e_j(n) = \exp(2\pi ijn/r)$ form an orthonormal system and the Fourier expansion of f is

$$f = \sum_{k=1}^{r} g(k)e_k, \quad \text{where} \quad g(k) = \langle f, e_k\rangle = \frac{1}{r}\sum_{j=1}^{r} f(j)\exp(-2\pi ijk/r),$$

which is Theorem 1.3, this is in fact the same proof as the proof of above.

Exercises

1.1. ▼ a) The **Cauchy product** of the functions $f, g : \mathbb{N}_0 = \{0, 1, 2, ...\} \to \mathbb{C}$ is given by

$$(f \otimes g)(n) = \sum_{k=0}^{n} f(k)g(n-k) = \sum_{k+\ell=n} f(k)g(\ell)$$

(where the sum has $n+1$ terms).

Show that the set $\mathbb{C}^{\mathbb{N}_0}$ of functions $f : \mathbb{N}_0 \to \mathbb{C}$ forms an integral domain with respect to addition and Cauchy product. When has a function an inverse with respect to the Cauchy product?

b) The Cauchy product of the periodic functions (mod r) f, g is defined by

$$(f \odot g)(n) = \sum_{k+\ell \,\equiv\, n \pmod{r}} f(k)g(\ell),$$

where the sum is over the solutions (mod r) of the congruence $k + \ell \equiv n \pmod{r}$, and the sum has r terms.

Observe that $f \odot g$ is also periodic (mod r). What are the properties of this operation?

1.2. ▼ Show that the Cauchy product of the exponential functions $e_s(k) = \exp(2\pi i sk/r)$ and $e_t(\ell) = \exp(2\pi i t\ell/r)$ is

$$\sum_{k+\ell \equiv n \pmod r} e_s(k)e_t(\ell) = \begin{cases} re_s(n) = r\exp(2\pi i sn/r), & \text{if } s \equiv t \pmod r, \\ 0, & \text{otherwise.} \end{cases}$$

2. Ramanujan sums

Ramanujan sums (S. Ramanujan, 1918) are defined by

$$c_r(n) = \sum_{\substack{k \pmod r \\ (k,r)=1}} \exp(2\pi i k n / r),$$

where k runs through a reduced system of residues $\pmod r$ and $r, n \in \mathbb{Z}$, $r \geq 1$.

Remarks. 1) The notation $c_r(n) = c(n, r)$ is also used.

2) If $k \equiv k' \pmod r$, then $\exp(2\pi i k n/r) = \exp(2\pi i k' n/r)$, hence this definition is correct, it does not depend on the choosed reduced system of residues.

3) $c_r(n)$ is the sum of n-th powers of the r-th primitive roots of unity.

4) If k runs through a complete system of residues $\pmod r$, then by Theorem 1.2,

$$\sum_{k \pmod r} \exp(2\pi i k n/r) = \begin{cases} r, & \text{if } r \mid n, \\ 0, & \text{if } r \nmid n, \end{cases} \tag{1}$$

5) For $n = 0$, $c_r(0) = \phi(r)$ is the Euler function.

6) For $n = 1$, $c_r(1)$ is the sum of the r-th primitive roots of unity, and this is exactly the Möbius function $\mu(r)$, as it can be seen from the next result.

Theorem 2.1. *For all $r, n \in \mathbb{Z}, r \geq 1$ we have*

$$c_r(n) = \sum_{d|(n,r)} d\mu(r/d).$$

Moreover, all the values of $c_r(n)$ are real integers, therefore

$$c_r(n) = \sum_{\substack{k \pmod r \\ (k,r)=1}} \cos(2\pi kn/r).$$

Proof. $c_r(n) = \sum_{k \pmod r} \exp(2\pi ikn/r) \sum_{d|(k,r)} \mu(d)$, where $d \mid (k,r) \Leftrightarrow d \mid k$ and $d \mid r$, and using the notation $k = dj$, $c_r(n) =$

$$= \sum_{d|r} \mu(d) \sum_{j \pmod{r/d}} \exp(2\pi ijn/(r/d)) = \sum_{d|r} \mu(d) \begin{cases} \frac{r}{d}, & \text{if } \frac{r}{d} \mid n \\ 0, & \text{otherwise} \end{cases} = \sum_{\substack{d|r \\ \frac{r}{d}|n}} \mu(d)\frac{r}{d},$$

using (1). Now let $\frac{r}{d} = \delta$. Then $c_r(n) = \sum_{\substack{\delta|n \\ \delta|r}} \delta\mu(\frac{r}{\delta}) = \sum_{\delta|(n,r)} \delta\mu(\frac{r}{\delta})$. ■

Remarks. 1. From Theorem 2.1 we have

$$\sum_{d|r} c_d(n) = \begin{cases} r, & \text{if } r \mid n, \\ 0, & \text{if } r \nmid n, \end{cases}$$

which can be shown also directly.

2. $|c_r(n)| \le \min(\sigma(n), \phi(r))$ for all $n, r \ge 1$, where $\sigma(n)$ is the sum of divisors of n and $\phi(r)$ is the Euler function.

Indeed, by Theorem 2.1, $|c_r(n)| \le \sum_{d|(n,r)} d \le \sum_{d|n} d = \sigma(n)$, and by definition $|c_r(n)| \le \phi(r) \le r$.

Theorem 2.2. *The function $c_r(n)$ is multiplicative in r, that is $c_{rs}(n) = c_r(n)c_s(n)$ for all $r, s \geq 1, (r, s) = 1$ (and for every fixed $n \geq 1$). Moreover*

$$c_{p^a}(n) = \begin{cases} p^a - p^{a-1}, & \text{if } p^a \mid n, \\ -p^{a-1}, & \text{if } p^a \nmid n, p^{a-1} \mid n, \\ 0, & \text{if } p^{a-1} \nmid n. \end{cases}$$

Proof. Let

$$I_r(n) = \begin{cases} r, & \text{if } r \mid n, \\ 0, & \text{if } r \nmid n, \end{cases}$$

which is multiplicative in r (for all $(r, s) = 1$ we have $rs \mid n \Leftrightarrow r \mid n, s \mid n$), and by Theorem 2.1,

$$c_r(n) = \sum_{d \mid r} I_d(n)\mu(\frac{r}{d}), \tag{2}$$

showing that $c_r(n)$ is multiplicative in r, being the convolution of two multiplicative functions ($c_\circ(n) = I_\circ(n) * \mu$). ■

Remark. $c_r(n)$ is not multiplicative in n. Indeed, if for example $p \neq q$ are primes, then $c_p(p) = p - 1$, $c_p(q) = -1$ and $c_p(pq) = p - 1 \neq -(p - 1) = c_p(p)c_p(q)$.

Theorem 2.3. *(Hölder identity) For all $n, r \geq 1$,*

$$c_r(n) = \frac{\phi(r)\mu(m)}{\phi(m)}, \qquad \text{where} \qquad m = \frac{r}{(n,r)}.$$

Proof. Both sides are multiplicative in r, hence it is enough to prove for $r = p^a$, a prime power (▼ Exercise!). ■

Theorem 2.4. *(Orthogonality relation) If $d \mid r$ and $t \mid r$, then*

$$\frac{1}{r}\sum_{m=1}^{r} c_d(m)c_t(m) = \begin{cases} \phi(d), & \text{if } d \neq t, \\ 0, & \text{if } d = t. \end{cases}$$

Proof. Let $r = dd_1$, $r = tt_1$. Then

$$\sum_{m=1}^{r} c_d(m)c_t(m) = \sum_{m=1}^{r} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \exp(2\pi i am/d) \sum_{\substack{b=1 \\ (b,t)=1}}^{t} \exp(2\pi i bm/t) =$$

$$= \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \sum_{\substack{b=1 \\ (b,t)=1}}^{t} \sum_{m=1}^{r} \exp(2\pi i m(a/d + b/t)),$$

where $a/d + b/t = (ad_1 + bt_1)/r$ and the inner sum, referring to m, is r if $r \mid (ad_1 + bt_1)$ and it is 0 otherwise (Theorem 1.2.) Here $1 \le d_1 \le ad_1 \le dd_1 = r$, $1 \le t_1 \le bt_1 \le tt_1 = r$, therefore $1 < ad_1 + bt_1 \le 2r$ and the equality is valid iff $d = t = 1$, since $(a, d) = (b, t) = 1$.

If $d > 1$ or $t > 1$, then $r \mid (ad_1 + bt_1) \Leftrightarrow ad_1 + bt_1 = r \Leftrightarrow add_1t + bdtt_1 = rdt \Leftrightarrow at + bd = dt \Leftrightarrow d = t$ and $a + b = d$. The proof of the last equivalence:

„$\Rightarrow$" $d \mid at, t \mid bd \Rightarrow d \mid t, t \mid d$ since $(a, d) = (b, t) = 1$, hence $d = t, a + b = d$. „$\Leftarrow$" is trivial. We obtain that for $d = t$ the sum of above is

$$r \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \sum_{\substack{b=1 \\ (b,d)=1}}^{d} \sum_{a+b=d} 1 = r\phi(d),$$

since there are $\phi(d)$ possible values of a, and for each there is exactly one value of b. ■

Theorem 2.5. *a) If* $r, s \in \mathbb{N}$, *then*

$$\sum_{n \le x} c_r(n) c_s(n) = \begin{cases} \phi(r)x + \mathcal{O}(1), & \text{if } r = s, \\ \mathcal{O}(1), & \text{if } r \neq s. \end{cases}$$

b) If $r \in \mathbb{N}$, *then*

$$\sum_{n \le x} c_r(n) = \begin{cases} x + \mathcal{O}(1), & \text{if } r = 1, \\ \mathcal{O}(1), & \text{if } r > 1. \end{cases}$$

Proof. a) Let r, s be fixed and $[x] = (rs)q + \alpha$, where $0 \le \alpha < rs$. Then

$$\sum_{n \le x} c_r(n) c_s(n) = \sum_{n=1}^{(rs)q} c_r(n) c_s(n) + \sum_{n=(rs)q+1}^{[x]} c_r(n) c_s(n) = \Sigma_1 + \Sigma_2.$$

Here $|\Sigma_2| \le \sum_{n=(rs)q+1}^{[x]} rs < (rs)^2$ is bounded in x and by Theorem 2.4 $\Sigma_1 = \phi(r)[x] = \phi(r)x + \mathcal{O}(1)$ for $r = s$, and $\Sigma_1 = 0$ for $r \neq s$.

b) Let $s = 1$, then $c_s(n) = 1$ for every n and apply Part a). ■

We say that an arithmetic function f has a **mean value** if the limit $M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$ exists.

According to Theorem 2.5 the product $c_r(n)c_s(n)$ and $c_r(n)$ possess a mean value given by

$$M(c_r c_s) = \begin{cases} \phi(r), & \text{if } r = s, \\ 0, & \text{if } r \neq s, \end{cases} \qquad M(c_r) = \begin{cases} 1, & \text{if } r = 1, \\ 0, & \text{if } r > 1. \end{cases}$$

The estimate of $\sum_{n\le x} c_r(n)$ can be given also by the first formula of Theorem 2.1. If $r = 1$, then $c_1(n) = 1$ and it is immediate that $\sum_{n\le x} c_1(n) = [x] = x + R(x)$, where $R(x) = -\{x\} \in (-1, 0]$. In case $r > 1$ we have the following result.

Theorem 2.6. *If $r > 1$ and $x \geq 1$, then $|\sum_{n\leq x} c_r(n)| \leq \psi(r)$, where $\psi(r) = r\prod_{p|r}(1 + 1/p)$ is the Dedekind function.*

Proof. Using the first formula of Theorem 2.1 we obtain

$$\sum_{n\leq x} c_r(n) = \sum_{\substack{n\leq x\\ d|(n,r)}} d\mu(r/d) = \sum_{d|r} d\mu(r/d) \sum_{n=dk\leq x} 1 = \sum_{d|r} d\mu(r/d)[x/d] =$$

$$= \sum_{d|r} d\mu(r/d)(x/d - \{x/d\}) = x\sum_{d|r} \mu(r/d) - \sum_{d|r} d\mu(r/d)\{x/d\} =$$

$$= -\sum_{d|r} d\mu(r/d)\{x/d\} = R_r(x),$$

where $|R_r(x)| \leq \sum_{d|r} d\mu^2(r/d) = \psi(r)$. ∎

Other orthogonality properties of Ramanujan sums are given in the following theorem.

Theorem 2.7. *(Orthogonality properties) If $r \mid k$ and $s \mid k$, then*

$$\sum_{d|k} c_r(k/d)c_d(k/s) = \begin{cases} k, & \text{if } r = s, \\ 0, & \text{if } r \neq s, \end{cases} \tag{3}$$

$$\sum_{d|k} \phi(d)c_r(k/d)c_s(k/d) = \begin{cases} k\phi(r), & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases} \tag{4}$$

Proof. We prove relation (3), where k/s has to be an integer, this is why we have condition $s \mid k$. The other condition $r \mid k$ will be used in the proof. By Theorem 2.1,

$$S := \sum_{d|k} c_r(k/d)c_d(k/s) = \sum_{d|k} c_d(k/s) \sum_{\delta|(r,k/d)} \delta\mu(r/\delta).$$

Let $r = \delta\ell, k/d = \delta m$, then

$$S = \sum_{\substack{\delta\ell=r \\ \delta dm=k}} \delta\mu(r/\delta)c_d(k/s) = \sum_{\delta|(r,k)} \delta\mu(r/\delta) \sum_{d|\frac{k}{\delta}} c_d(k/s),$$

where $(r,k)=r$ and the inner sum is $\frac{k}{\delta}$ for $\frac{k}{\delta} \mid \frac{k}{s} \Leftrightarrow s \mid \delta$ and it is 0 otherwise. Cosequently,

$$S = \sum_{\substack{\delta \mid r \\ s \mid \delta}} \delta \mu(r/\delta) \frac{k}{\delta} = k \sum_{\substack{\delta \mid r \\ s \mid \delta}} \mu(r/\delta),$$

here for each term $s \mid r$, therefore $S=0$ if $s \nmid r$. If $s \mid r$, then denoting $r=\delta u, \delta = st$,

$$S = k \sum_{stu=r} \mu(\frac{r}{st}) = k \sum_{t \mid \frac{r}{s}} \mu(\frac{r/s}{t}) = \begin{cases} k, & \text{if } r/s = 1, \\ 0, & \text{otherwise.} \end{cases}$$

To prove (4) we need the following: If $r \mid k$, $s \mid k$, then

$$\phi(r) c_s(k/r) = \phi(s) c_r(k/s). \tag{5}$$

This is true, since by the Hölder identity (Theorem 2.3),

$$\phi(r) c_s(k/r) = \frac{\phi(r)\phi(s)\mu(\frac{s}{(s,k/r)})}{\phi(\frac{s}{(s,k/r)})}, \qquad \phi(s) c_s(k/s) = \frac{\phi(s)\phi(r)\mu(\frac{r}{(r,k/s)})}{\phi(\frac{r}{(r,k/s)})},$$

where $\dfrac{s}{(s,k/r)} = \dfrac{r}{(r,k/s)}$. Now by (5) and (3),

$$\sum_{d|k} \phi(d) c_r(k/d) c_s(k/d) = \sum_{d|k} \phi(s) c_d(k/s) c_r(k/d) =$$

$$= \phi(s) \sum_{d|k} c_r(k/d) c_d(k/s) = \begin{cases} k\phi(r), & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases} \quad \blacksquare$$

Theorem 2.8. *The Dirichlet series of the Ramanujan sum* $c_r(n)$*, as a function in* r*, is*

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{r^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}, \quad n \geq 1, \operatorname{Re} s > 1,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ *and* ζ *is the Riemann zeta function.*

Proof. $c_r(n)$ is bounded in r, since $|c_r(n)| \leq \sigma(n)$ (see above), the series is absolutely convergent for $\operatorname{Re} s > 1$ and $c_r(n) = (I_n * \mu)(r)$, where $I_n(d) = d$ for $d \mid n$ and $I_n(d) = 0$ for $d \nmid n$ (Theorem 2.2). We obtain

$$\sum_{r=1}^{\infty} \frac{c_r(n)}{r^s} = \sum_{r=1}^{\infty} \frac{I_n(r)}{r^s} \sum_{r=1}^{\infty} \frac{\mu(r)}{r^s} = \sum_{\substack{r=1 \\ r|n}}^{\infty} \frac{r}{r^s} \cdot \frac{1}{\zeta(s)} =$$

$$= \frac{1}{\zeta(s)} \sum_{r|n} r^{1-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}. \blacksquare$$

Remarks. 1. For $s = 2$ we obtain

$$\frac{\sigma(n)}{n} = \frac{\pi^2}{6} \sum_{r=1}^{\infty} \frac{c_r(n)}{r^2} = \frac{\pi^2}{6} \left(1 + \frac{(-1)^n}{2^2} + \frac{2\cos(2\pi n/3)}{3^2} + \frac{2\cos(\pi n/2)}{4^2} + \cdots \right),$$

where $\sigma(n) = \sum_{d|n} d$, showing how the values of $\sigma(n)/n$ fluctuate harmonically about their mean value $\pi^2/6$.

2. A Fourier analysis of arithmetical functions, with respect to Ramanujan sums, parallel to periodic and almost periodic functions, was developed by G. H. Hardy (1921), E. Cohen (1960), J. Knopfmacher (1975), A. Hildebrand (1984), W. Schwarz, J. Spilker(1994) and others.

They studied expansions, convergent pointwise or in other sense, of arithmetic functions f of the form

$$f(n) = \sum_{r=1}^{\infty} a_r c_r(n), \quad n \in \mathbb{N},$$

where the Ramanujan coefficients a_r are

$$a_r = \frac{1}{\phi(r)} M(f c_r).$$

3. It can be shown that for $s = 1$, $\displaystyle\sum_{r=1}^{\infty} \frac{c_r(n)}{r} = 0$. In this case $\sum_{r=1}^{\infty} \frac{\mu(r)}{r} = 0$ is convergent, but not absolutely convergent (this is equivalent with the prime number theorem).

Exercises

2.1. ▼ Show that if $n \geq 1$, $d \mid r$ and $e \mid r$, then the Cauchy product of $c_d(k)$ and $c_e(\ell)$ is (see Exercises 1.1., 1.2.):

$$\sum_{k+\ell \,\equiv\, n \pmod{r}} c_d(k)c_e(\ell) = \begin{cases} rc_d(n), & \text{if } d = e, \\ 0, & \text{if } d \neq e. \end{cases}$$

2.2. ▼ Show that the Dirichlet series of $c_r(n)$, as a function in n, is

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \zeta(s)\phi_{1-s}(r), \quad r \geq 1, \operatorname{Re} s > 1,$$

where $\phi_t(r) = \sum_{d|r} d^t \mu(r/d)$ is the generalized Euler function.

3. Even functions

The function $f : \mathbb{N} \to \mathbb{C}$ is called **even function** (mod r) if $f(n) = f((n, r))$ for all n, that is if the value $f(n)$ depends only on the gcd (n, r). Hence if f is even (mod r), then it is sufficient to know the values $f(d)$, where $d \mid r$.

Every function f which is even (mod r) is periodic (mod r), since $f(n + r) = f((n+r, r)) = f((n, r)) = f(n)$ for all n. For example the functions $f(n) = (n, r)$ and $f(n) = c_r(n)$ are even functions (mod r).

Question: How can even functions (mod r) be characterized?

We show that every even function (mod r) is a linear combination of Ramanujan sums. More exactly,

Theorem 3.1. *If f is an even function (mod r), then f can be written in the form*

$$f(n) = \sum_{q \mid r} h(q) c_q(n), \quad n \in \mathbb{N},$$

where the values $h(q)$ are uniquely determined (Fourier coefficients of f) and

$$h(q) = \frac{1}{r\phi(q)} \sum_{e \mid r} \phi(e) f(r/e) c_q(r/e) = \frac{1}{r} \sum_{e \mid r} f(r/e) c_e(r/q), \quad q \mid r.$$

Proof. Similar to the proof of Theorem 1.3., using the orthogonality properties of above.

Another way: Let $\mathcal{E}_r$ denote the set of even functions (mod r), which is a complex linear space with the addition of functions and poitwise multiplication. Furthermore $\mathcal{E}_r$ is isomorphic to the space $\mathbb{C}^{\tau(r)}$, where $\tau(r)$ is the number of divisors of r. Hence $\mathcal{E}_r$ is a Hilbert space of dimension $\tau(r)$ and

$$\langle f, g\rangle = \frac{1}{r}\sum_{d|r} \phi(d) f(r/d)\overline{g(r/d)}$$

is an inner product, $\phi(d)$ denoting the Euler function, where $\overline{g(r/d)}$ is the complex conjugate of $g(r/d)$.

The functions $c'_q(n) = \frac{1}{\sqrt{\phi(q)}} c_q(n)$, where $q \mid r$, form an orthonormal basis. Indeed, if $q \mid r$, $t \mid r$, then by (4),

$$\langle c'_q, c'_t\rangle = \frac{1}{r}\sum_{d|r} \phi(d) c'_q(r/d) c'_t(r/d) =$$

$$= \frac{1}{r\sqrt{\phi(q)\phi(t)}}\sum_{d|r} \phi(d) c_q(r/d) c_t(r/d) = \begin{cases} 1, & \text{if } q = t, \\ 0, & \text{if } q \neq t. \end{cases}$$

We obtain that the Fourier expansion of f according to c'_q $(q \mid r)$ is

$$f(n) = \sum_{q|r} j(q) c'_q(n), \quad n \in \mathbb{N},$$

where

$$j(q) = \langle f, c'_q \rangle = \frac{1}{r} \sum_{e|r} \phi(e) f(r/e) c'_q(r/e) = \frac{1}{r\sqrt{\phi(q)}} \sum_{e|r} \phi(e) f(r/e) c_q(r/e).$$

Therefore

$$h(q) = \frac{1}{\sqrt{\phi(q)}} j(q) = \frac{1}{r\phi(q)} \sum_{e|r} \phi(e) f(r/e) c_q(r/e).$$

Using also (5),

$$h(q) = \frac{1}{r\phi(q)} \sum_{e|r} \phi(q) f(r/e) c_e(r/q) = \frac{1}{r} \sum_{e|r} f(r/e) c_e(r/q),$$

which was to be proved. ■

Theorem 3.2. (L. Tóth, 2004) *If f is an even function (mod r), then there exists the main value $M(f)$ of f and it is given by*

$$M(f) = \frac{1}{r}(f * \phi)(r).$$

Proof. According to Theorem 3.1, f can be written as $f(n) = \sum_{q|r} h(q)c_q(n)$, $n \in \mathbb{N}$, where

$$M(f) = \sum_{q|r} h(q)M(c_q) = h(1) = \frac{1}{r}\sum_{e|r} \phi(e)f(r/e)c_1(r/e) =$$

$$= \frac{1}{r}\sum_{e|r} \phi(e)f(r/e) = \frac{1}{r}(f * \phi)(r). \blacksquare$$

Exercises

3.1. ▼ Let f, g be even functions (mod r) with Fourier coefficients $\alpha(q)$ and $\beta(q)$, respectively. Prove that the Cauchy product $f \odot g$ is also an even function (mod r) having Fourier coefficients $r\alpha(q)\beta(q)$.

3.2. ▼ If f is an even function (mod r), then according to Theorem 3.2, f has a main value $M(f)$ and $M(f) = \frac{1}{r}(f*\phi)(r)$. Estimate the difference $\sum_{n\le x} f(n) - xM(f)$.

3.3. ▼ Let $(s,d)=1$ and let $\phi(s,d,n)$ denote the number of elements in the arithmetic progression $s, s+d, ..., s+(n-1)d$ which are coprime to n. This is a generalization of the Euler function, $\phi(1,1,n)=\phi(n)$ is the Euler function. Prove that

i) $\phi(s,d,n)$ is multiplicative in n,

ii) for every prime power p^k, $\phi(s,d,p^k)=\begin{cases} p^k(1-1/p), & (p,d)=1, \\ p^k, & p \mid d, \end{cases}$

(these values do not depend on s),

iii) $\displaystyle\sum_{n\le x}\phi(s,d,n)=\frac{3d^2}{\pi^2 J(d)}x^2+\mathcal{O}(x\log x)$, where $J(d)=\phi_2(d)=d^2\prod_{p|d}(1-1/p^2)$ is the Jordan function,

iv) $f(d):=\phi(s,d,n)=\dfrac{\phi(n)(n,d)}{\phi((n,d))}$ and this is an even function (mod n) in d,

v) $\displaystyle M(f)=n\prod_{p|n}(1-1/p+1/p^2)$.

Solutions of the exercises

1.1. a) The **Cauchy product** of the functions $f, g : \mathbb{N}_0 = \{0, 1, 2, \ldots\} \to \mathbb{C}$ is given by

$$(f \otimes g)(n) = \sum_{k=0}^{n} f(k)g(n-k) = \sum_{k+\ell=n} f(k)g(\ell)$$

(where the sum has $n+1$ terms).

Show that the set $\mathbb{C}^{\mathbb{N}_0}$ of functions $f : \mathbb{N}_0 \to \mathbb{C}$ forms an integral domain with respect to addition and Cauchy product. When has a function an inverse with respect to the Cauchy product?

b) The Cauchy product of the periodic functions (mod r) f, g is defined by

$$(f \odot g)(n) = \sum_{k+\ell \equiv n \pmod r} f(k)g(\ell),$$

where the sum is over the solutions (mod r) of the congruence $k + \ell \equiv n \pmod r$, and the sum has r terms.

Observe that $f \odot g$ is also periodic (mod r). What are the properties of this operation?

Solution. a) It is immediate that $(\mathbb{C}^{\mathbb{N}_0}, +)$ is an abelian group, $\otimes$ is commutative, distributive with respect to addition and $\varepsilon(0) = 1, \varepsilon(n) = 0, n > 0$ is the identity element. Furthermore,

$$((f \otimes g) \otimes h)(n) = (f \otimes (g \otimes h))(n) = \sum_{k+\ell+m=n} f(k)g(\ell)h(m),$$

the operation is associative. There are no divisors of zero: let $f, g \neq 0$, where $f(a) \neq 0, f(b) \neq 0$, and a, b are the least numbers with this property. Then

$$(f \otimes g)(a+b) = f(a)g(b) + \sum_{\substack{k+\ell=a+b \\ k\neq a, \ell\neq b}} f(k)g(\ell) = f(a)g(b) \neq 0,$$

since for $k < a$ one has $f(k) = 0$, and if $k > a$, then $\ell < b$ and $g(\ell) = 0$.

f has an inverse $\overline{f}$ iff $f(0) \neq 0$ and in this case

$$\overline{f}(0) = \frac{1}{f(0)}, \qquad \overline{f}(n) = -\frac{1}{f(0)} \sum_{k=1}^{n} f(k)\overline{f}(n-k), \; n > 0.$$

b) $\odot$ is commutative, associative, distributive to addition. There is no identity element, since it would be $\varepsilon(0) = 1, \varepsilon(n) = 0$, $n > 0$, but this is not periodic.

If $\mathcal{P}_r$ denotes the set of periodic functions (mod r), then $(\mathcal{P}_r, +, \odot)$ is a commutative ring.

Remark. If $f, g \in \mathcal{P}_r$, then in general $f \otimes g \neq f \odot g$.

1.2. Show that the Cauchy product of the exponential functions $e_s(k) = \exp(2\pi i s k/r)$ and $e_t(\ell) = \exp(2\pi i t\ell/r)$ is

$$\sum_{k+\ell \equiv n \pmod r} e_s(k)e_t(\ell) = \begin{cases} r e_s(n) = r\exp(2\pi i s n/r), & \text{if } s \equiv t \pmod r, \\ 0, & \text{otherwise.} \end{cases}$$

Solution.

$$\sum_{k+\ell \equiv n \pmod r} \exp(2\pi i k s/r)\exp(2\pi i \ell t/r) =$$

$$= \sum_{\substack{k \pmod r \\ \ell \equiv n-k \pmod r}} \exp(2\pi i k s/r)\exp(2\pi i (n-k)t/r) =$$

$$= \exp(2\pi i n t/r) \sum_{k \pmod r} \exp(2\pi i k(s-t)/r),$$

which is $r\exp(2\pi i n s/r) = r\exp(2\pi i n t/r)$ if $s \equiv t \pmod r$ and 0 otherwise, see Theorem 1.2.

2.1. Show that if $n \geq 1$, $d \mid r$ and $e \mid r$, then the Cauchy product of $c_d(k)$ and $c_e(\ell)$ is (see Exercises 1.1., 1.2.):

$$\sum_{k+\ell \equiv n \pmod r} c_d(k)c_e(\ell) = \begin{cases} rc_d(n), & \text{if } d = e, \\ 0, & \text{if } d \neq e. \end{cases}$$

Solution. Let $r = dd_1$, $r = ee_1$. Then

$$\sum_{k+\ell \equiv n \pmod r} c_d(k)c_e(\ell) = \sum_{k+\ell \equiv n \pmod r} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \exp(2\pi i ak/d) \sum_{\substack{b=1 \\ (b,e)=1}}^{e} \exp(2\pi i b\ell/e) =$$

$$= \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \sum_{\substack{b=1 \\ (b,e)=1}}^{e} \sum_{k+\ell \equiv n \pmod r} \exp(2\pi i akd_1/r) \exp(2\pi i b\ell e_1/r) =$$

$$= \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \sum_{\substack{b=1 \\ (b,e)=1}}^{e} \sum_{ad_1 \equiv be_1 \pmod r} r \exp(2\pi i ad_1 n/r),$$

by Exercise 1.2. Here $1 \leq d_1 \leq ad_1 \leq dd_1 = r$, $1 \leq e_1 \leq be_1 \leq ee_1 = r$, hence $ad_1 \equiv be_1 \pmod r$ valid iff $ad_1 = be_1 \Leftrightarrow ad_1de = be_1de \Leftrightarrow are = bde \Leftrightarrow ae = bd$.

But $(a,d)=1, (b,e)=1$, therefore $a \mid b, b \mid a$, that is $a=b, d=e$ and obtain that in this case the sum is

$$r \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \exp(2\pi i a n/d) = r c_d(n).$$

If $d \neq e$, then each term is zero and the sum is also zero.

2.2. Show that the Dirichlet series of $c_r(n)$, as a function in n, is

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \zeta(s)\phi_{1-s}(r), \quad r \geq 1, \operatorname{Re} s > 1,$$

where $\phi_t(r) = \sum_{d|r} d^t \mu(r/d)$ is the generalized Euler function.

Solution. $|c_r(n)| \leq r$ is bounded as a function in n, the series is absolutely convergent for $\operatorname{Re} s > 1$ and by Theorem 2.1,

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \sum_{n=1}^{\infty} \Big(\sum_{d|(n,r)} d\mu(r/d) \Big) \frac{1}{n^s} = \sum_{d|r} d\mu(r/d) \sum_{\delta=1}^{\infty} \frac{1}{(d\delta)^s} =$$

$$= \sum_{d|r} d^{1-s} \mu(r/d) \sum_{\delta=1}^{\infty} \frac{1}{\delta^s} = \zeta(s)\phi_{1-s}(r).$$

3.1. ▼ Let f, g be even functions (mod r) with Fourier coefficients $\alpha(q)$ and $\beta(q)$, respectively. Prove that the Cauchy product $f \odot g$ is also an even function (mod r) having Fourier coefficients $r\alpha(q)\beta(q)$.

Solution.

$$h(n) = \sum_{k+\ell \equiv n \pmod r} f(k)g(\ell) = \sum_{k+\ell \equiv n \pmod r} \sum_{q|r} \alpha(q)c_q(k) \sum_{s|r} \beta(s)c_s(\ell) =$$

$$= \sum_{q|r} \sum_{s|r} \alpha(q)\beta(s) \sum_{k+\ell \equiv n \pmod r} c_q(k)c_s(\ell) = \sum_{q|r} \sum_{s|r} \alpha(q)\beta(s) \sum_{q=s} rc_q(n) =$$

$$= \sum_{q|r} r\alpha(q)\beta(q)c_q(n),$$

using the result of Exercise 3.1. Here $c_r(n)$ is an even function (mod r) and for $q \mid r$ the function $c_q(n)$ is also even (mod r), hence $h(n)$ is even (mod r) and the Fourier coefficients of $h(n)$ are exactly $r\alpha(q)\beta(q)$.

3.2. ▼ If f is an even function (mod r), then according to Theorem 3.2, f has a main value $M(f)$ and $M(f) = \frac{1}{r}(f * \phi)(r)$. Estimate the difference $\sum_{n \le x} f(n) - xM(f)$.

Solution. Let $R_f(x) = \sum_{n \le x} f(n) - xM(f)$. By Theorem 3.2 and Theorem 2.5./ b) with the notations $C_1 = 1, C_q = 0, q > 1$,

$$\sum_{n \le x} f(n) = \sum_{n \le x} \sum_{q|r} h(q) c_q(n) = \sum_{q|r} h(q) \sum_{n \le x} c_q(n) = \sum_{q|r} h(q)(C_q x + R_q(x)) =$$

$$= h(1)x + \sum_{q|r} h(q) R_q(x),$$

where $h(1) = \frac{1}{r}(f * \phi)(r) = M(f)$, $|h(q)| \le \frac{1}{r} \sum_{e|r} |f(r/e)||c_e(r/q)| \le$
$\le \frac{1}{r} \sum_{e|r} |f(r/e)| e = \sum_{e|r} \frac{|f(e)|}{e}$ and using also Theorem 2.6,

$$|R_f(x)| = |\sum_{q|r} h(q) R_q(x)| \le \sum_{e|r} \frac{|f(e)|}{e} \sum_{q|r} |R_q(x)| \le \sum_{e|r} \frac{|f(e)|}{e} \sum_{q|r} \psi(q) =$$

$$= rF(r) \sum_{e|r} \frac{|f(e)|}{e},$$

where $F(r) = \sum_{d|r} \psi(d) = r \prod_{p|r} (1 + 2(1 - p^{-k})(p-1)^{-1})$.

Remark. f is periodic, hence bounded, let $|f(n)| \le K, n \ge 1$. Then $|R_f(x)| \le KF(r)\sigma(r)/r$. It is true that for all $\varepsilon > 0$, $|R_f(x)| \le KC_\varepsilon r^{1+\varepsilon}$, where C_ε depends only on ε, see the paper of L. TÓTH, 2004, Proposition 1.

3.3. ▼ Let $(s,d)=1$ and let $\phi(s,d,n)$ denote the number of elements in the arithmetic progression $s, s+d, ..., s+(n-1)d$ which are coprime to n. This is a generalization of the Euler function, $\phi(1,1,n)=\phi(n)$ is the Euler function. Prove that

i) $\phi(s,d,n)$ is multiplicative in n,

ii) for every prime power p^k, $\phi(s,d,p^k)=\begin{cases} p^k(1-1/p), & (p,d)=1, \\ p^k, & p \mid d, \end{cases}$

(these values do not depend on s),

iii) $\sum_{n\le x}\phi(s,d,n)=\frac{3d^2}{\pi^2 J(d)}x^2+\mathcal{O}(x\log x)$, where $J(d)=\phi_2(d)=d^2\prod_{p|d}(1-1/p^2)$ is the Jordan function,

iv) $f(d):=\phi(s,d,n)=\frac{\phi(n)(n,d)}{\phi((n,d))}$ and this is an even function (mod n) in d,

v) $M(f)=n\prod_{p|n}(1-1/p+1/p^2)$.

Solution. More generally, let $f \in \mathbb{Z}[X]$ be a polynomial with integer coefficients and let $\phi_f(n)$ denote the number of integers $x \pmod n$ such that $(f(x), n) = 1$. If $f = s + (x-1)d$, then $\phi_f(n) = \phi(s, d, n)$ for all n.

i) We show that ϕ_f is multiplicative. Let $N_f(n)$ denote the number of solutions $\pmod n$ of the congruence $f(x) \equiv 0 \pmod n$. Using the properties of the Möbius function,

$$\phi_f(n) = \sum_{\substack{x=1 \\ (f(x),n)=1}}^{n} 1 = \sum_{x=1}^{n} \sum_{e \mid (f(x),n)} \mu(e) = \sum_{x=1}^{n} \sum_{\substack{e \mid f(x) \\ e \mid n}} \mu(e) =$$

$$= \sum_{e \mid n} \mu(e) \sum_{\substack{1 \le x \le n \\ f(x) \equiv 0 \pmod e}} 1 = \sum_{e \mid n} \mu(e) \frac{n}{e} N_f(e).$$

Therefore $\phi_f = \mu N_f * E$, and since $N_f(n)$ is multiplicative in n we obtain that ϕ_f is multiplicative too.

ii) According to this convolutional identity we have for every prime power p^k the relation (*) $\phi_f(p^k) = p^k - p^{k-1} N_f(p) = p^k(1 - N_f(n)/p)$.

Let $f = s + (x-1)d$. How many solutions has the congruence $s + (x-1)d \equiv 0 \pmod{p^k}$? This is equivalent to $dx \equiv d - s \pmod{p^k}$, being a linear congruence.

If $(p,d) = 1$, then $(p^k, d) = 1$ and obtain that $N_f(p^k) = 1$ for every k. If $(p,d) \neq 1$, then $(p^k, d) = p^a$, $a \geq 1$, and since $(s,d) = 1$ we obtain that $p \nmid d - s$, consequently $N_f(p^k) = 0$. Use now relation (*).

iii) For the function $\phi(s,d,n)$ using that N_f is bounded,

$$\sum_{n \leq x} \phi(s,d,n) = \sum_{ab \leq x} \mu(a) N_f(a) b = \sum_{a \leq x} \mu(a) N_f(a) \sum_{b \leq x/a} b =$$

$$= \sum_{a \leq x} \mu(a) N_f(a) \Big(\frac{x^2}{2a^2} + \mathcal{O}(\frac{x}{a})\Big) =$$

$$= \frac{x^2}{2} \sum_{a=1}^{\infty} \frac{\mu(a) N_f(a)}{a^2} + \mathcal{O}(x^2 \sum_{a > x} \frac{1}{a^2}) + \mathcal{O}(x \sum_{a \leq x} \frac{1}{a}) =$$

$$= \frac{x^2}{2} \prod_p \Big(1 - \frac{N_f(p)}{p^2}\Big) + \mathcal{O}(x) + \mathcal{O}(x \log x) = \frac{x^2}{2} C + \mathcal{O}(x \log x),$$

where $C = \prod_{p \nmid d} \Big(1 - \frac{1}{p^2}\Big) = \prod_p \Big(1 - \frac{1}{p^2}\Big) \prod_{p \mid d} \Big(1 - \frac{1}{p^2}\Big)^{-1} = \frac{1}{\zeta(2)} \cdot \frac{d^2}{\phi_2(d)}$.

iv) By the multiplicativity and ii) we obtain at once:

$$\phi(s,d,n) = \frac{\phi(n)(n,d)}{\phi((n,d))} = \phi(n) \cdot \frac{(n,d)}{\phi((n,d))},$$

where the variable is d, $\phi(n)$ is fixed and $\frac{(n,d)}{\phi((n,d))}$ depends only on the gcd (n,d) and by definition it is an even function $(\bmod n)$.

v) Use Theorem 3.2 for $r = n$.

Remark. The function $\phi(s,d,n)$ was investigated by P. G. Garcia and S. Ligh (1983, 1985), see also the papers of L. Tóth (1987, 1990).

The function $\phi_f(n)$ was investigated by P. K. Menon (1967), and by H. Stevens (1971). The asymptotic formula for $\phi_f(n)$ was established by L. Tóth and J. Sándor (1989). The results of iv) and v) were proved by T. Maxsein (1990).

References

P. J. McCarthy, Introduction to Arithmetical Functions, Springer Verlag, New York Berlin Heidelberg Tokyo, 1986.

R. Sivaramakrishnan, Classical Theory of Arithmetic Functions, Marcel Dekker, New York-Basel, 1989.

W. Schwarz, J. Spilker, Arithmetical Functions, London Math. Soc. Lecture Notes Series **184**, Cambridge Univ. Press, 1994.

P. Haukkanen, An elementary linear algebraic approach to even functions (mod r), Nieuw Arch. Wiskd. (5), **2** (2001), 29-31.

P. G. Garcia, S. Ligh, A generalization of Euler's ϕ-function, Fibonacci Quart., **21** (1983), 26-28.

S. Ligh, P. G. Garcia, A generalization of Euler's ϕ-function, II., Math. Japonica, **30** (1985), 519-522.

P. Kesava Menon, An extension of Euler's function, Math. Student, **35** (1967), 55-59.

H. Stevens, Generalizations of the Euler ϕ-function, Duke Math. J., **38** (1971), 181-186.

T. Maxsein, A note on a generalization of Euler's ϕ-function, Indian J. Pure Appl. Math., **21** (1990), 691-694.

L. Tóth, A note on a generalization of Euler's function, Fibonacci Quart., **25** (1987), 241-244.

L. Tóth, J. Sándor, An asymptotic formula concerning a generalized Euler function, Fibonacci Quart., **27** (1989), 176-180.

L. Tóth, Some remarks on a generalization of Euler's function, Seminar Arghiriade 23 (1990), 9pp.

L. Tóth, Remarks on generalized Ramanujan sums and even functions, Acta Math. Acad. Paedagog Nyházi. (N. S.), electronic, **20** (2004), 233-238, see

http://de.arxiv.org/PS_cache/math/pdf/0610/0610358.pdf