Proposed problems - Kitűzött feladatok

• Newsletter of the European Mathematical Society, Issue 127, March 2023, Problem 273 (Proposed by László Tóth):

Let $c_n(k)$ denote the Ramanujan sum defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integer $m \geq 1$,

$$\sum_{[n,k]=m} c_n(k) = \varphi(m),$$

where the sum is over all ordered pairs (n, k) of positive integers n, k such that their lcm is m, and φ is Euler's totient function.

• Newsletter of the European Mathematical Society, Issue 127, March 2023, Problem 274 (Proposed by László Tóth):

Show that, for every integer $n \geq 1$, we have the polynomial identity

$$\prod_{\substack{k=1\\(k,n)=1}}^{n} \left(x^{(k-1,n)} - 1\right) = \prod_{d|n} \Phi_d(x)^{\varphi(n)/\varphi(d)},$$

where $\Phi_d(x)$ are the cyclotomic polynomials and φ denotes Euler's totient function.

• Newsletter of the European Mathematical Society, Issue 103, March 2017, Problem 172 (Proposed by László Tóth):

Show that for every integer $n \geq 1$ and every real number $a \geq 1$ one has

$$\frac{1}{2n} \leq \frac{1}{n^{a+1}} \sum_{k=1}^{n} k^a - \frac{1}{a+1} < \frac{1}{2n}\left(1 + \frac{1}{2n}\right)^a.$$

• Newsletter of the European Mathematical Society, Issue 103, March 2017, Problem 173 (Proposed by László Tóth):

Let $c_n(k)$ denote the Ramanujan sum defined as the sum of kth powers of the primitive nth roots of unity. Show that for any integers n, k, a with $n \geq 1$,

$$\sum_{d|n} c_d(k) a^{n/d} \equiv 0 \pmod{n}.$$

• American Mathematical Monthly, vol. 118, 2011, Problem 11576 (Proposed by László Tóth):

Let $\omega(n)$ denote the number of distinct prime factors of n. Let $P(x,k)$ be the set of integers in $[1,x]$ that are relatively prime to k, and let $\phi(x,k) = |P(x,k)|$. Let

$$S(x,k) = \sum_{n \in P(x,k)} (-1)^{\omega(n)}.$$

Show that for all real x in $[1,\infty)$,

$$\sum_{1 \le n \le x} (-1)^{\omega(n)} \phi(x/n, n) = \sum_{1 \le n \le x} S(x/n, n) = 1.$$

• American Mathematical Monthly, vol. 98, 1991, Problem E 3432 (Proposed by László Tóth):

(i) Prove that for every positive integer n we have

$$\frac{1}{2n+2/5} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n - \gamma < \frac{1}{2n+1/3},$$

where γ is Euler's constant.

(ii) Show that $2/5$ can be replaced by a slightly smaller number, but that $1/3$ cannot be replaced by a slightly larger number.

• American Mathematical Monthly, vol. 94, 1987, Problem E 3211 (Proposed by László Tóth):

Let $\omega(k)$ denote the number of distinct prime factors of the positive integer k and let (i,n) denote the greatest common divisor of the positive integers i and n. Express

$$\sum_{i=1}^{n} 2^{\omega((i,n))}$$

in terms of the prime factorization of n.

• Matematikai Lapok (Kolozsvár) 2/1992 szám, 22620. Feladat, XI. osztály (Kitűzte Tóth László):

Legyen $x_n = \dfrac{(n+1)^{n-1}}{n^{n+1}}$, $(\forall)\ n \ge 1$. Számítsuk ki a

$$\lim_{n \to \infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right)$$

határértéket.

• Matematikai Lapok (Kolozsvár) 1/1991 szám, 22255. Feladat, XI. osztály (Kitűzte Tóth László):

Alkalmazva a Wallis-képletet (lásd 21842. feladat, ML 7/1989. sz.) igazoljuk, hogy

$$\sqrt{\pi\left(n+\frac{1}{4}\right)} < \frac{(2n)!!}{(2n-1)!!} < \sqrt{\pi\left(n+\frac{1}{3}\right)}, \quad (\forall)\ n \ge 1.$$

• Matematikai Lapok (Kolozsvár) 1-2/1990 szám, 22035. Feladat, XI. osztály (Kitűzte Tóth László):

Az $(x_n)_{n\geq 1}$ sorozatot a következő összefüggéssel értelmezzük:

$$\left(1+\frac{1}{n-x_n}\right)^n = e, \quad (\forall)\ n \geq 1.$$

Igazoljuk, hogy a sorozat szigorúan növekvő, korlátos és $\lim\limits_{n\to\infty} x_n = \dfrac{1}{2}$. (A ML. 3/1985 számának 20382. feladatával kapcsolatban.)

• Matematikai Lapok (Kolozsvár) 5-6/1988 szám, 21463. Feladat, XI. osztály (Kitűzte Tóth László):

Igazoljuk, hogy

$$\ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{1}{n+1}\right) > \ln\left(1+\frac{1}{n}\right)\cdot\ln\left(1+\frac{1}{n+1}\right), \quad (\forall)\ n\in\mathbb{N}, n\geq 1.$$